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Sub Manifolds of a Riemannian Manifold Admitting a Quarter Symmetric Semi Metric Connection

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Abstract

Semi symmetric metric connections have been studied by Imai[1]. Mishra and Pandey [3] defined the notion of quarter symmetric connection in a differentiable manifold. A semi symmetric semi metric connection has been introduced by Barua and Mukhopadhyay. In the present paper submanifolds of a Riemannian manifold has been considered which admits quarter symmetric semi metric connection. Gauss and Weingarten formulae for such a connection have been derived. Gauss, Codazzi and Ricci equations for such connection have been deduced. Finally, we have studied hypersurfaces of Riemannian manifold admitting a quarter symmetric semi metric connection and established Gauss and Codazzi equations.

Keywords

Quarter symmetric semi metric, Riemannian manifold

Preliminaries

Let M^n be an n–dimensional differentiable manifold of class C^∞ immersed in the Riemannian manifold M^{n+p}. Let the enveloping manifold M^{n+p} a Riemannian manifold with metric tensor $\bar{g}$. Then the submanifold M^n is also a Riemannian manifold with induced metric tensor g defined by

(1.1) $$\bar{g}(\bar{X}, \bar{Y}) = g(X, Y).$$

Let $\xi_1, \xi_2, \ldots, \xi_p$ be unit vectors in M^{n+p} which are normal to M^n at every point on it, then

(1.2) $$\bar{g}(X, \xi_i) = 0, \quad i = 1, 2, 3, \ldots, p.$$

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The extension of X, Y, Z, … on M^{n+p} may be denoted by $\bar{X}$, $\bar{Y}$, $\bar{Z}$,….

If $\bar{\nabla}$ is the Riemannian connection on M^{n+p}, then by Gauss formula [6]

(1.3) $$\bar{\nabla}_Y Z = \nabla_Y Z + h(Y, Z)$$

where ∇ is the induced Riemannian connection on M^n and h(Y, Z) is the second fundamental form, given by

(1.4) $$h(Y, Z) = b^i(Y, Z)\xi_i \;; \quad i = 1,2,3,\ldots,p$$

$b^i(Y, Z)$ being the second fundamental tensors corresponding to the ξ_i. The Weingarten formula is given by

$$\bar{\nabla}_Y \xi_i = -A_i Y + \nabla^{\perp}_Y \xi_i \;; \quad i = 1, 2, 3, \ldots, p$$

where A_i is the Weingarten map with respect to ξ_i and $\nabla^{\perp}$ is the Riemannian connection on $T^{\perp}(M^n)$. It is known that [6]

(1.6) $$g(A_i Y, Z) = b^i(Y, Z).$$

From (1.4) and (1.5)

(1.7) $$\bar{R}(X, Y)Z = R(X, Y)Z - \{b^i(Y, Z)A_i X - b^i(X, Z)A_i Y\} + \{\nabla_X b^i(Y, Z) - \nabla_Y b^i(X, Z)\}\xi_i + \{b^i(Y, Z)\nabla^{\perp}_X \xi_i - b^i(X, Z)\nabla^{\perp}_Y \xi_i\}.$$

Thus, for any vector field U on M^n,

(1.8) $$\bar{K}(X,Y,Z,U) = K(X,Y,Z,U) - \sum_{i=1}^{p}\{b^i(Y,Z)b^i(X,U) - b^i(X,Z)\,b^i(Y,U)\}$$

where K(X,Y,Z,U) = g(R(X, Y)Z, U) is the fully covariant Riemannian curvature tensor on M^n and $\bar{K}$(X,Y,Z,U) is the same on M^{n+p}. This is Gauss equation[6].

The normal component of $\bar{R}$(X, Y)Z is

(1.9) $$(\bar{R}(X,Y)Z)^n = \{(\nabla_X b^i)(Y,Z) - (\nabla_Y b^i)(X, Z)\}\xi_i + \{b^i(Y, Z)\nabla_X^{\perp}\xi_i - b^i(X, Z)\nabla_Y^{\perp}\xi_i\}.$$

Hence for $\eta \in T^{\perp}(M^n)$

(1.10) $$(\bar{K}(X, Y, Z, \eta))^n = \{(\nabla_X b^i)(Y, Z) - (\nabla_Y b^i)(X, Z)\}\,\bar{g}(\xi_i, \eta)$$

$$+ \{b^i(Y, Z)\overline{g}(\nabla^{\perp}_X \xi_i, \eta) - b^i(X, Z)\overline{g}(\nabla^{\perp}_Y \xi_i, \eta)\}.$$

This is Codazzi equation.

If η, ζ are two vector fields normal to M^n, then the Ricci equation is

$$\overline{K}(X, Y, \eta, \zeta) = K^n(X, Y, \eta, \zeta) + \overline{g}([A_\zeta, A_\eta](X), Y)$$

where

$$K^n(X, Y, \eta, \zeta) = \overline{g}(R^n(X, Y)\eta, \zeta),$$

$$R^n(X, Y)\eta = \nabla^{\perp}_X \nabla^{\perp}_Y \eta - \nabla^{\perp}_Y \nabla^{\perp}_X \eta - \nabla^{\perp}_{[X, Y]} \eta,$$

is the curvature tensor on $T^{\perp}(M^n)$ and $[A_\zeta, A_\eta] = A_\zeta A_\eta - A_\eta A_\zeta$.

Gauss and Weingarten Formula for a Quarter Symmetric Semi Metric Connection

Let M^{n+p} admits quarter symmetric semi metric connection $\overline{D}$ defined by [2]

(2.1) $$\overline{D}_{\overline{Y}}\overline{Z} = \overline{\nabla}_{\overline{Y}}\overline{Z} - \overline{\pi}(\overline{Y})\overline{F}(\overline{Z}) + \overline{g}(\overline{F}(\overline{Y}), \overline{Z})\overline{P},$$

where $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are vector fields on M^{n+p}, $\overline{P}$ is the associated vector field and $\overline{\pi}$ is the associated differential 1–form defined by

(2.2) $$\overline{g}(\overline{P}, \overline{Y}) = \overline{\pi}(\overline{Y}).$$

$\overline{\nabla}_{\overline{Y}}$ denotes the Levi-Civita connection with respect to Riemannian metric $\overline{g}$. F is a tensor of type (1, 1).

If the Riemannian manifold M^{n+p} and M^n are orientable, we can choose p vector fields N_i defined along M^n such that

$$\overline{g}(BFX, N_i) = 0$$

and

$$\overline{g}(N_i, N_j) = \delta_i^j$$

for arbitrary vector field X in M^n. We call this vector field the unit normal vector field to the submanifold M^n. Also

$$\overline{g}(\overline{F}(\overline{X}), Y) = \overline{g}(\overline{X}, \overline{F}(\overline{Y})),$$

(2.3) $$(\overline{D}_{\overline{X}}\overline{g})(\overline{Y}, \overline{Z}) = 2\overline{\pi}(\overline{X})\overline{g}(\overline{Y}, \overline{F}(\overline{Z})) - \overline{\pi}(\overline{Y})\overline{g}(\overline{F}(\overline{X}), \overline{Z}) - \overline{\pi}(\overline{Z})\overline{g}(\overline{Y}, \overline{F}(\overline{X})).$$

Writing

(2.4) $\bar{P} = P + a^i\xi_i$; i = 1, 2, …., p

where P is tangential to M^n, we can easily find

(2.5) $\bar{D}_Y Z = D_Y Z + m^i(Y, Z)\xi_i$,

(2.6) $D_Y Z = \nabla_Y Z - \pi(Y)F(Z) + g(F(Y), Z)P$

and π is the restriction of $\bar{\pi}$ on M^n satisfying

(2.7) $g(P, X) = \pi(X)$.

Further

(2.8) $m^i(Y, Z) = b^i(Y, Z) + a^i g(FY, Z)$

is the second fundamental tensor with respect to $\bar{D}$ corresponding to ξ_i. From the equation (2.6) it follows that D is a quarter symmetric semi metric connection on M^n induced by $\bar{D}$, where

(2.9) $(D_X g)(Y, Z) = 2\pi(X)g(Y,F(Z)) - \pi(Y)g(F(X),Z) - \pi(Z)g(Y, F(X))$.

Hence (2.5) is the Gauss formula for the quarter symmetric semi metric connection $\bar{D}$ on M^{n+p}.

Let

(2.10) $\bar{D}_X\eta = -M_\eta X + D^{\perp}_X \eta$,

where η is vector field normal to M^n. $M_\eta X$ is tangential to M^n and $D^{\perp}_X \eta$ is normal to it. It can be seen by straight forward calculation that M_n is a linear map of $T(M^n)$ which depends linearly on η and $D^{\perp}$ is a derivative on

$T^{\perp}(M^{\eta})$.

If η, ζ are vector fields normal to T (M^n) then

$$(\bar{D}_X \bar{g})(\eta, \zeta) = \bar{D}_X(\bar{g}(\eta, \zeta)) - \bar{g}(\bar{D}_X\eta, \zeta) - \bar{g}(\eta, \bar{D}_X\zeta)$$

$$= \bar{D}_X(\bar{g}(\eta, \zeta)) - \bar{g}(D^{\perp}_X \eta, \zeta) - \bar{g}(\eta, D^{\perp}_X \zeta)$$

$$= (D^{\perp}_X \bar{g})(\eta, \zeta)$$

and by equation (1.2) and (2.3)

$$(\bar{D}_X \bar{g})(\eta, \zeta) = 2\bar{\pi}(X)\bar{g}(\eta, \bar{F}(\zeta)).$$

Hence,

$$(D^{\perp}_X \bar{g})(\eta, \zeta) = 2\bar{\pi}(X)\bar{g}(\eta, \bar{F}(\zeta)).$$

THEOREM 2.1

If $\overline{D}$ is a quarter symmetric semi metric connection on a Riemannian manifold M^{n+p} then on a submanifold $M^n \subset M^{n+p}$ the Gauss formula and the Weingarten formula for $\overline{D}$ are given by (2.5) and (2.10).

THEOREM 2.2

The induced connection $D^{\perp}$ on $T^{\perp}(M^n)$ is a recurrent metric connection with $\overline{\pi}$ as the 1–form of recurrence.

Now,

$$(\overline{D}_X g)(Y, \eta) = -\overline{\pi}(\eta)\,\overline{g}(FX, Y).$$

Again

(2.8) $$(\overline{D}_X \overline{g})(Y, \eta) = \overline{D}_X \overline{g}(Y, \eta) - \overline{g}(\overline{D}_X Y, \eta) - \overline{g}(Y, \overline{D}_X \eta).$$

Using (1.2), (2.5), (2.7) and (2.8)

$$(\overline{D}_X \overline{g})(Y, \eta) = -\overline{g}(m^i(X, Y)\xi_i, \eta) - \overline{g}(Y, -M_n X).$$

Hence

$$\overline{g}(M_n X, Y) = m^i(X, Y)\,\overline{g}(\xi_i, \eta) - \overline{\pi}\, g(FX, Y).$$

If in particular $\eta = \xi_k$, then

$$g(M_k X, Y) = m^k(X, Y) - a^k g(FX, Y),$$

since $\overline{g}(\xi_i, \xi_k) = \delta_{ik}$ and $\overline{\pi}(\xi_k) = a^k$. This is the relation between M_k and m^k.

GAUSS, CODAZZI AND RICCI EQUATION FOR THE CONNECTION $\overline{D}$

From (2.1) we get

$$_{\overline{D}}\overline{R}(\overline{X}, \overline{Y})\overline{Z} = \overline{R}(\overline{X}, \overline{Y})\overline{Z} - 2d\overline{\pi}(\overline{X}, \overline{Y})F\overline{Z} + \overline{g}(F\overline{Y}, \overline{Z})\overline{L}\overline{X} - \overline{g}(F\overline{X}, \overline{Z})\overline{L}\overline{Y}$$

$$_{\overline{D}}\overline{R}(\overline{X}, \overline{Y})\overline{Z} = \overline{D}_{\overline{X}}\overline{D}_{\overline{Y}}\overline{Z} - \overline{D}_{\overline{Y}}\overline{D}_{\overline{X}}\overline{Z} - \overline{D}_{[\overline{X}, \overline{Y}]}\overline{Z}$$

is the curvature tensor for the connection $\overline{D}$ and $\overline{R}(\overline{X}, \overline{Y})\overline{Z}$ is the Riemannian curvature tensor on M^{n+p}. Also

$$\overline{L}\overline{X} = \overline{\nabla}_{\overline{X}}\overline{P} + \overline{\pi}(\overline{X})\overline{P}.$$

If,

$$_{\bar{D}}\bar{K}(\bar{X},\bar{Y},\bar{Z},\bar{U}) = \bar{g}(\,_{\bar{D}}\bar{R}(\bar{X},\bar{Y})\bar{Z},\bar{U}),$$

then

$$_{\bar{D}}\bar{K}(\bar{X},\bar{Y},\bar{Z},\bar{U}) = \bar{K}(\bar{X},\bar{Y},\bar{Z},\bar{U}) - 2d\bar{\pi}(\bar{X},\bar{Y})\bar{g}(\bar{F}\bar{Z},\bar{U}) \tag{3.1}$$

$$+\ \bar{g}(\bar{F}\bar{Y},\ \bar{Z})\bar{\lambda}(\bar{X},\bar{U})\ -\ \bar{g}(\bar{F}\bar{X},\bar{Z})\ \bar{\lambda}(\bar{Y},\bar{U})$$

where $\bar{\lambda}(\bar{X},\bar{U}) = \bar{g}(\bar{L}\bar{X},\bar{U}) = (\bar{\nabla}_{\bar{X}}\bar{\pi})(\bar{U}) + \bar{\pi}(\bar{F}\bar{X})\bar{\pi}(\bar{U})$.

From Gauss and Weingarten formula (2.5) and (2.10) for the connection $\bar{D}$, we find

$$_{\bar{D}}\bar{R}\,(X,Y)Z = {}_{D}R(X, Y)Z + \{m^i(X, Y)M_iY - m^i(Y,\ Z)M_iX\}$$

$$+ \{(D_Xm^i)(Y, Z) - (D_Ym^i)(X, Z) - \pi(X)m^i(FY, Z)$$

$$+\pi(Y)m^i(FX, Z)\}\xi_i + \{m^i(Y, Z)\,D_X^{\perp}\xi_i - m^i(X, Z)\,D_Y^{\perp}\xi_i\}.$$

Hence,

$$_{\bar{D}}\bar{K}\,(X, Y, Z, U) = {}_{D}K(X, Y, Z, U) + \sum_{i=1}^{p}\{m^i(X, Z)m^i(Y, U)$$

$$- m^i(Y, Z)m^i(X, U)\} - \sum_{i=1}^{p} a^i\,\{m^i(X, Z)g(FY, U)$$

$$- m^i(Y, Z)g(FX, U)\}.$$

This is Gauss equation for the connection $\bar{D}$.

For a normal vector field η

$$_{\bar{D}}\bar{K}\,(X, Y, Z, \eta) = \{(D_Xm^i)(Y, Z) - D_Ym^i(X, Z)\}\,\bar{g}\,(\xi_i, \eta)$$

$$-\{\pi(X)m^i(FY, Z) - \pi(Y)m^i(FX, Z)\}\,\bar{g}\,(\xi_i, \eta)$$

$$+ m^i(Y, Z)\,\bar{g}\,(D_X^{\perp}\xi_i, \eta) - m^i(X, Z)\,\bar{g}\,(D_Y^{\perp}\xi_i, \eta).$$

If in particular $\eta = \xi_k$, then

$$_{\bar{D}}\bar{K}\,(X, Y, Z, \xi_k) = (D_Xm^k)(Y, Z) - D_Ym^k(X, Z) - \pi(X)m^k(FY, Z)$$

$$+ \pi(Y)m^k(FX, Z) + \{m^i(Y, Z)\,\bar{g}\,(D_X^{\perp}\xi_i, \xi_k)$$

$$- m^i(X, Z)\,\bar{g}\,(D_Y^{\perp}\xi_i, \xi_k).$$

This is codazzi equation for the connection $\bar{D}$.

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If

$$_{D}R^{\perp}(X, Y)\eta = D^{\perp}_{X} D^{\perp}_{Y}\eta - D^{\perp}_{Y} D^{\perp}_{X}\eta - D^{\perp}_{[X,Y]}\eta$$

and

$$_{D}K^{\perp}(X, Y, \eta, \zeta) = g(_{D}R^{\perp}(X, Y)\eta, \zeta),$$

then the Ricci equation is

$$_{\bar{D}}\bar{K}(X, Y, \eta, \zeta) = {}_{D}K^{\perp}(X, Y, \eta, \zeta) + g([M_{\eta}, M_{\zeta}]X, Y)$$

where

$$[M_{\eta}, M_{\zeta}] = M_{\eta}M_{\zeta} - M_{\zeta}M_{\eta}.$$

For the sake of simplicity, let us now make M^{n} as a hypersurface in M^{n+1}(p = 1). The Gauss and Wiengarten formula for the connection $\bar{D}$ are

$$\bar{D}_{Y}Z = D_{Y}Z + m(Y, Z)\xi,$$

$$\bar{D}_{Y}\xi = -M_{\xi}Y + D^{\perp}_{Y}\xi,$$

where ξ is the unit vector field normal to M^{n}.

Gauss and Codazzi equations takes the simplier forms, namely

(3.2) $$_{\bar{D}}\bar{K}(X,Y,Z,U) = {}_{D}K(X, Y, Z, U) + m(Y, U)m(X, Z) - m(X, U)m(Y,Z) - a\{g(FY, U)m(X, Z) - g(FX, U)m(Y, Z)\}$$

and

$$_{\bar{D}}\bar{K}(X, Y, Z, \xi) = D_{X}m(Y, Z) - D_{Y}m(X, Z) - \{\pi(X)m(FY, Z) - \pi(Y)m(FX, Z)\}.$$

Also from (2.4) and (2.8) we get

(3.3) $$\bar{P} = p + a\xi,$$

(3.4) $$m(Y, Z) = b(Y, Z) + ag(FY, Z).$$

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